Pergamon

PII: S0021-8928(96)00090-1

J. Appl. Maths Mechs, Vol. 60, No. 5, pp. 713–716, 1996

0021-8928/96 $24.00+0.00

CONSTRUCTION OF AN AIRCRAFT STABILIZING CONTROL USING THE LYAPUNOV FUNCTION†

D. I. BUGROV

Moscow

(*Received* 14 *December* 1994)

The complete non-linear equations of motion of an aircraft, treated as an absolutely rigid body, are considered. The control parameters are the engine thrust and the angles of deviation of the rudders. The problem of stabilizing a programmed regime of the aircraft motion is formulated. The regime is described by specifying the variation with time of the velocity vectors of the mass centre and the angular velocity. The so-called "kinetic energy of the deviations" is taken as a Lyapunov function, and a stabilizing control is written explicitly. It is proved that the control ensures asymptotic stability of "practically" any programmed regime of motion. © 1997 Elsevier Science Ltd. All rights reserved.

The dynamical equations of motion of an aircraft as an absolutely rigid body with constant mass and constant inertia tensor, expressed in a system of coordinates xyz attached to the centre of mass of the aircraft, with the x axis pointing along the body to the nose, the y axis pointing upward and the z axis along the right wing, are as follows:

$$m\frac{d\mathbf{v}}{dt}+\boldsymbol{\omega}\times\mathbf{v}=\mathbf{F}+m\mathbf{g},\qquad J\frac{d\boldsymbol{\omega}}{dt}+\boldsymbol{\omega}\times J\boldsymbol{\omega}=\mathbf{M} \tag{1}$$

where m is the mass of the aircraft, $\mathbf{v}$ is the absolute velocity vector of its centre of mass in projections onto the xyz axes, $\boldsymbol{\omega}$ is the absolute angular velocity vector of the aircraft in projections onto the same axes, J is the inertia tensor of the craft relative to the xyz areas, t is the flight time, $\mathbf{g}$ is the acceleration due gravity, $\mathbf{F}$ is the total vector of non-gravitational forces applied to the aircraft (aerodynamic forces and the thrust force $\mathbf{P}$ of the engine, which is constant in direction), and $\mathbf{M}$ is the moment of the aerodynamic forces.

It is assumed that the acceleration due to gravity is constant in magnitude and direction, the atmospheric density is constant, and the aerodynamic forces and moments acting on the aircraft depend only on the magnitude of the velocity vector $\mathbf{v}$, on the angle of attack and the glancing angle, which determine the position of $\mathbf{v}$ relative to the system of coordinates xyz, and also on the thrust P of the engine and the positions of the rudders (e.g. the direction rudder, elevator and ailerons), which form a vector $\boldsymbol{\delta}=(\delta_1,\delta_2,\delta_3)^T$ [1]. The magnitude of the controls in this formulation are subject to constraints

$$|\delta_i|\leqslant H_i\quad i=1,2,3;\quad 0\leqslant P\leqslant H_p \tag{2}$$

Suppose given a programmed regime, of the form

$$\mathbf{v}=\mathbf{v}^p(t),\qquad \boldsymbol{\omega}=\boldsymbol{\omega}^p(t) \tag{3}$$

which satisfies Eqs (1) with an open-loop control $\mathbf{P}^p$, $\boldsymbol{\delta}^p$ satisfying the constraints (2). Our aim is to solve the stabilization problem for this regime.

As new variables, we will consider the deviations from the regime (3)

$$\Delta\mathbf{v}=\mathbf{v}-\mathbf{v}^p(t),\qquad \Delta\boldsymbol{\omega}=\boldsymbol{\omega}-\boldsymbol{\omega}^p(t)$$

in the region $\Gamma=\{\|\Delta\mathbf{v}\|\leqslant V,\|\Delta\boldsymbol{\omega}\|\leqslant\Omega\}$. Then the equations of motion (1) may be rewritten in the form‡

†*Prikl. Mat. Mekh.* Vol. 60, No. 5, pp. 719–723, 1996.

‡BYUSHGENS G S., GOMAN M. G., ZAGAINOV G. I., MATYUKHIKN V. L. and PYATNITSKII Ye. S., The method of Lyapunov functions in problems of synthesizing the control of the space motion of an aircraft. Preprint, Institute of Control Problems, Moscow, 1992.

$$m\left(\frac{d\Delta\mathbf{v}}{dt}\right)+\Delta\boldsymbol{\omega}\times\Delta\mathbf{v}+\Delta\boldsymbol{\omega}\times\mathbf{v}^p(t)+\boldsymbol{\omega}^p(t)\times\Delta\mathbf{v}=\Delta\mathbf{F}$$
$$J\frac{d\Delta\boldsymbol{\omega}}{dt}+\Delta\boldsymbol{\omega}\times J\Delta\boldsymbol{\omega}+\Delta\boldsymbol{\omega}\times J\boldsymbol{\omega}^p(t)+\boldsymbol{\omega}^p(t)\times J\Delta\boldsymbol{\omega}=\Delta\mathbf{M} \tag{4}$$

where

$$\Delta\mathbf{F}=\mathbf{F}(\mathbf{v}^p+\Delta\mathbf{v},\boldsymbol{\omega}^p+\Delta\boldsymbol{\omega},\boldsymbol{\delta}^p+\Delta\boldsymbol{\delta},\mathbf{P}^p+\Delta\mathbf{P},t)-\mathbf{F}(\mathbf{v}^p,\boldsymbol{\omega}^p,\boldsymbol{\delta}^p,\mathbf{P}^p,t)$$
$$\Delta\mathbf{M}=\mathbf{M}(\mathbf{v}^p+\Delta\mathbf{v},\boldsymbol{\omega}^p+\Delta\boldsymbol{\omega},\boldsymbol{\delta}^p+\Delta\boldsymbol{\delta},\mathbf{P}^p+\Delta\mathbf{P},t)-\mathbf{M}(\mathbf{v}^p,\boldsymbol{\omega}^p,\boldsymbol{\delta}^p,\mathbf{P}^p,t) \tag{5}$$

To calculate an additional control $\Delta\boldsymbol{\delta}$, $\Delta\mathbf{P}$ guaranteeing asymptotic stability of the zero solution of Eqs (4), we use a Lyapunov function. Such a function will be a modified "kinetic energy of deviations"†

$$G=\frac{1}{2}\Delta\boldsymbol{\omega}\cdot J\Delta\boldsymbol{\omega}+\frac{1}{2}Rm\Delta\mathbf{v}\cdot\Delta\mathbf{v} \tag{6}$$

where R is a positive weighting factor, and $G(\cdot)$ is a positive-definite quadratic form. The stabilizing control is constructed so as to ensure that the derivative of the Lyapunov function along trajectories of the system of deviational equations will always be negative

$$\dot G(\boldsymbol{\delta},\mathbf{P},\boldsymbol{\omega},\mathbf{v})<0 \tag{7}$$

The total derivative of the Lyapunov function (6) along trajectories of Eqs (4) is

$$\dot G=\Delta\boldsymbol{\omega}\cdot J\Delta\boldsymbol{\omega}+Rm\Delta\mathbf{v}\cdot\Delta\mathbf{v}=Rm\mathbf{v}^p\cdot(\Delta\boldsymbol{\omega}\times\Delta\mathbf{v})+$$
$$+\boldsymbol{\omega}^p\cdot(\Delta\boldsymbol{\omega}\times J\Delta\boldsymbol{\omega})+Rm\Delta\mathbf{v}\cdot\Delta\mathbf{F}+\Delta\boldsymbol{\omega}\cdot\Delta\mathbf{M}$$

Conditions (7) will hold if

$$\dot G=Rm\mathbf{v}^p\cdot(\Delta\boldsymbol{\omega}\times\Delta\mathbf{v})+\boldsymbol{\omega}^p\cdot(\Delta\boldsymbol{\omega}\times J\Delta\boldsymbol{\omega})+$$
$$+Rm\Delta\mathbf{v}\cdot\Delta\mathbf{F}(\boldsymbol{\delta},\mathbf{P},\ldots)+\Delta\boldsymbol{\omega}\cdot\Delta\mathbf{M}(\boldsymbol{\delta},\mathbf{P},\ldots)<0 \tag{8}$$

Thus, our initial problem of synthesizing a stabilizing control has been reduced† to the algebraic problem of solving inequality (8) for the unknown controls $\boldsymbol{\delta}$ and $\mathbf{P}$. There is no general analytic solution of inequality (8), because $\Delta\mathbf{F}$ and $\Delta\mathbf{M}$ are non-linear functions of the parameter of motion! The weakest assumption which enables one to solve Eq. (8) that is also valid for practically all aircraft [1] is that the aerodynamic forces and moments are linear functions of the positions of the rudders, and that the thrust of the engine is directed strictly along the x axis. In that case the last two terms of Eq. (8) are replaced by

$$Rm\Delta\mathbf{v}\cdot\left(\Delta\tilde J+\frac{\partial\mathbf{F}}{\partial\boldsymbol{\delta}}\Delta\boldsymbol{\delta}\right)+Rm\Delta\mathbf{v}_x\Delta P+\Delta\boldsymbol{\omega}\cdot\left(\Delta\tilde{\mathbf{M}}+\frac{\partial\mathbf{M}}{\partial\boldsymbol{\delta}}\Delta\boldsymbol{\delta}\right)$$

where $\Delta\tilde{\mathbf{F}}$ and $\Delta\tilde{\mathbf{M}}$ differ from expressions (5) in that the controls are not varied.

The solution of problem (8) may be expressed in the form

$$\Delta\boldsymbol{\delta}=-\left(\frac{\partial\mathbf{M}}{\partial\boldsymbol{\delta}}\right)^{-1}[K\Delta\boldsymbol{\omega}+L_0],\qquad \mathbf{L}_0=Rm\Delta\mathbf{v}\times\mathbf{v}^p+(J\Delta\boldsymbol{\omega})\times\boldsymbol{\omega}^p+\Delta\tilde{\mathbf{M}} \tag{9}$$

$$\Delta P=-k_4\Delta\mathbf{v}_x-d_0,\qquad d_0=\Delta\mathbf{v}\cdot\Delta\tilde{\mathbf{F}}+\Delta\mathbf{v}\cdot\frac{\partial\mathbf{F}}{\partial\boldsymbol{\delta}}\Delta\boldsymbol{\delta} \tag{10}$$

†BYUSHGENS G. S., GOMAN M. G., ZAGAINOV G. I., MATYUKHIKN V. L. and PYATNITSKII Ye. S., The method of Lyapunov functions in problems of synthesizing the control of the space motion of an aircraft. Preprint, Institute of Control Problems, Moscow, 1992.

where K is a diagonal matrix with positive elements $k_1, k_2, k_3; k_4 > 0$. That the matrix $\partial \mathbf{M}/\partial \boldsymbol{\delta}$, calculated along the programmed regime (2), is invertible follows from the fact that the aircraft is controllable relative to its centre of mass (see footnote on previous page). The value of $\Delta\boldsymbol{\delta}$ appearing in (10) is calculated in advance using formula (9).

Note that when the solution (9), (10) is substituted into (8), only the non-strict inequality $\dot{G} \leqslant 0$ is true, i.e. one can only prove that the zero solution of Eqs (4) is stable in Lyapunov's sense.

To verify that the zero solution of the unsteady-state equations (4), closed by the control (9), (10), is asymptotically stable, one can use a criterion for the derivative of the Lyapunov function to be sign-definite (see [2], Theorem 1.1). Consider the function $W = (\Delta\mathbf{v} \times J\Delta\boldsymbol{\omega})_x$, i.e. the first coordinate of this vector product. The function W is bounded on Γ, $\dot{G} = 0$ on $E = \{\Delta\boldsymbol{\omega} = 0, \Delta \mathbf{v}_x = 0\}$. Evaluating the derivative of W along trajectories of Eqs (4), (9) and (10) on E, we get

$$\dot{W}\Big|_{\substack{\Delta\omega=0\\ \Delta v_x=0}} = \left[\Delta\mathbf{v}\times J\frac{d\Delta\boldsymbol{\omega}}{dt}\Big|_{\substack{\Delta\omega=0\\ \Delta v_x=0}} + \frac{d\Delta\mathbf{v}}{dt}\times d\Delta\boldsymbol{\omega}\Big|_{\substack{\Delta\omega=0\\ \Delta v_x=0}}\right]_x =$$

$$= \left[\Delta\mathbf{v}\times\left[\Delta\tilde{\mathbf{M}} + \frac{\partial\mathbf{M}}{\partial\boldsymbol{\delta}}\left(-\frac{\partial\mathbf{M}}{\partial\boldsymbol{\delta}}\right)^{-1}[K\Delta\boldsymbol{\omega} + Rm\Delta\mathbf{v}\times\mathbf{v}^p + (J\Delta\boldsymbol{\omega})\times\boldsymbol{\omega}^p + \Delta\tilde{\mathbf{M}}] - \right.\right.$$

$$\left.\left. -\Delta\boldsymbol{\omega}\times J\Delta\boldsymbol{\omega} - \Delta\boldsymbol{\omega}\times J\boldsymbol{\omega}^p - \boldsymbol{\omega}^p\times J\Delta\boldsymbol{\omega}\right] + \frac{d\Delta\mathbf{v}}{dt}\times J\Delta\boldsymbol{\omega}\Big|_{\substack{\Delta\omega=0\\ \Delta v_x=0}}\right]_x =$$

$$= [\Delta\mathbf{v}\times(-K\Delta\boldsymbol{\omega} - Rm\Delta\mathbf{v}\times\mathbf{v}^p - \Delta\boldsymbol{\omega}\times J\Delta\boldsymbol{\omega} - \Delta\boldsymbol{\omega}\times J\boldsymbol{\omega}^p) - \Delta\boldsymbol{\omega}\times J\boldsymbol{\omega}^p) +$$

$$\left. + \frac{d\Delta\mathbf{v}}{dt}\times d\Delta\boldsymbol{\omega}\Big|_{\substack{\Delta\omega=0\\ \Delta v_x=0}}\right]_x = [-Rm\Delta\mathbf{v}\times[\Delta\mathbf{v}\times\mathbf{v}^p]]_x = Rm\mathbf{v}_x^p(\Delta\mathbf{v}_y^2 + \Delta\mathbf{v}_z^2)$$

$$R > 0, \quad m > 0$$

Thus, for all programmed trajectories except $\mathbf{v}_x^p = 0$ (but in that case such important characteristics as the angle of attack and the glancing angle are undefined), one can use the standard definition [2, Definition 1.1] to define $\dot{W} \neq 0$ in the set E. The remaining conditions of a Theorem 1.1 of [2] are satisfied since G and $\dot{G}$ do not depend explicitly on the time.

The control just determined, in the form (9), (10), requires the minimum possible assumptions concerning the aerodynamic characteristics of the aircraft; it implies that the motion (3) is asymptotically stable (provided, of course, that constraints (2) are obeyed).

We present the results of a numerical solution of the stabilization problem for the three-dimensional trajectory of an aircraft subject to the control (9), (10). The experiment was carried out using a model of a highly manoeuvrable aircraft with the following characteristics: mass m = 19 tons, characteristic area S = 56.6 m^2, mean aerodynamic chord b_a = 3.8 m, wing span l = 13.05 m, and inertia tensor—diagonal with components $J_x = 2.608 \times 10^4$ kg m^2, $J_y = 1.6067 \times 10^5$ kg m^2, $J_z = 1.378 \times 10^5$ kg m^2. The aerodynamic forces and moments were taken to be the following functions of the flight conditions

$$\mathbf{F}_a = \mathbf{F} - \mathbf{P} = (-X, Y, Z), \quad \mathbf{M} = (M_x, M_y, M_z)$$

$$X = c_x\frac{\rho S v^2}{2}, \quad Y = c_y\frac{\rho S v^2}{2}, \quad Z = c_z\frac{\rho S v^2}{2}$$

$$M_x = m_x\frac{\rho S v^2}{2}l, \quad M_y = m_y\frac{\rho S v^2}{2}l, \quad M_z = m_z\frac{\rho S v^2}{2}b_a$$

$$c_x = c_{x1}(\alpha) + c_{x2}(\alpha)\delta_1, \quad c_y = c_{y1}(\alpha) + c_{y2}(\alpha)\delta_1, \quad c_z = c_{z1}(\alpha)\beta + c_{z2}(\alpha)\delta_2$$

$$m_x = m_{x1}(\alpha)\beta + m_{x2}(\alpha)\delta_3 + m_{x3}(\alpha)\delta_2 + m_{x4}(\alpha)\omega_x + m_{x5}(\alpha)\omega_y$$

$$m_y = m_{y1}(\alpha)\beta + m_{y2}(\alpha)\delta_3 + m_{y3}(\alpha)\delta_2 + m_{y4}(\alpha)\omega_y + m_{y5}(\alpha)\omega_x$$

$$m_z = m_{z1}(\alpha) + m_{z2}(\alpha)\delta_1 + m_{z3}(\alpha)\omega_z$$

where ρ is the density of the oncoming flow, $c_x, c_y, c_z, m_x, m_y, m_z$ are the non-dimensional coefficients of the respective forces and moments, and α and β and are the angle of attack and glancing angle.

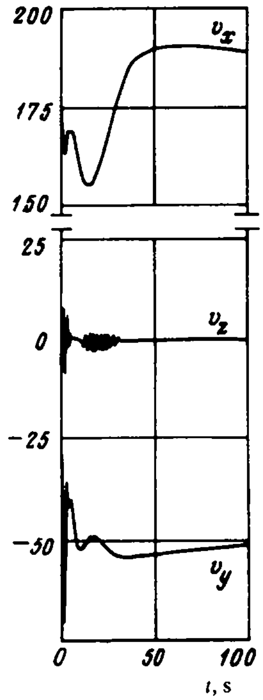

Fig. 1.

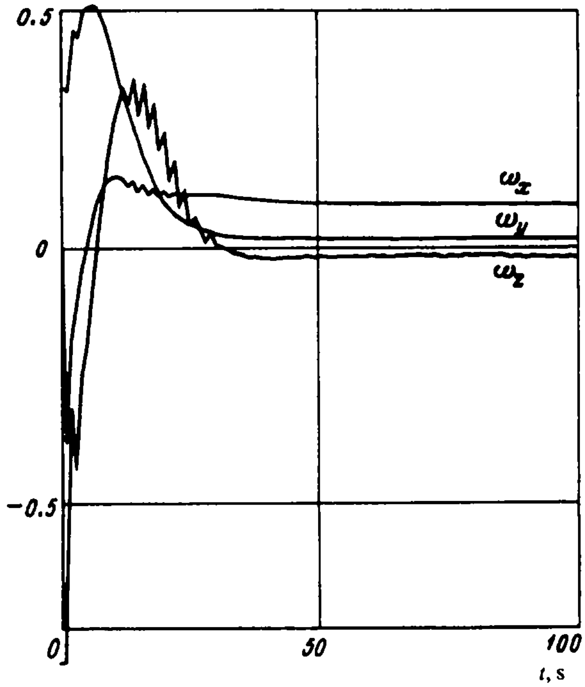

Fig. 2.

The simulation made use of tabulated values for c_x, c_y, c_z, m_x, m_y, m_z, as listed in [3], with linear interpolation. The parameters occurring in (9) and (10) were determined as follows: $R = 10^{-3}$, $k_1 = k_2 = k_3 = k_4 = 10^6$.

The programmed trajectory sampled was descent at a constant velocity down a spiral, without slip (each turn was assumed to take 60 s). The results—the projections v_x, v_y and v_z of the velocity of the aircraft's centre of mass (m/s) and the projections ω_x, ω_y and ω_z of its absolute angular velocity [rad/s]—are shown in Figs 1 and 2. At the end of the transient all variables tend to stationary values corresponding to the selected programmed trajectory.

The method described for stabilizing arbitrary unsteady trajectories of an aircraft is utilized in the Imitation Model of Controlled Flight,† set up at the Mechanico-Mathematical Faculty, Moscow State University under the supervision of V. V. Aleksandrov. I wish to thank him and S. S. Lemak for useful comments on this research.

This work was carried out with financial support from the Russian Foundation for Basic Research (93-01-16241).

REFERENCES

1. BYUSHGENS G. S. and STUDNEV R. V., *Dynamics of the Three-dimensional Motion of an Aircraft*. Mashinostroyeniye, Moscow, 1967.
2. MATROSOV V. M., The stability of motion. *Prikl. Mat. Mekh.* **26**, 885–895, 1962.
3. BUKOV V. N., *Adaptive Predictive Systems of Flight Control*. Nauka, Moscow, 1987.

Translated by D.L.

†SADOVNICHII V. A., ALEKSANDROV V. V. *et al.*, Software for automated equipment for the teaching, training and testing of specialists in the control of dynamical systems, Preprint No. 3, Mechanico-Mathematical Faculty, Moscow State University, 1994.